On Conditional Monte Carlo in Rare Event Probability Estimation

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Talk Outline

1. **Introduction**
   - The Model Under Analysis
   - Standard Simulation
   - Conditional Monte Carlo
2. **Conditional MC on a Markov Chain**
   - Pure Conditional MC – Exact Calculation
   - Conditional MC – Intermediate Estimations
3. **Experimental Setting**
   - Model 1
   - Model 2
4. **Concluding Remarks**

Reliability Model
Markov Chain

- X: continuous time Markov Chain that model a highly reliable multi-component system; state space: S.
- Y: discrete time Markov chain, canonically embedded in X.
- $S = U \cup D \mid$ in U the system is up, in D the system is down.
- The system starts at $\mathbf{u} \in U$, and eventually comes back to $\mathbf{u}$ in time $\tau_{\mathbf{u}}$.
- D is collapsed in a single state $\mathbf{d}$, made absorbing.
- The system eventually hits $\mathbf{d}$ in time $\tau_{\mathbf{d}}$
- It is of interest the estimation of γ:

$$\gamma = \mathbb{P}\{\tau_{\mathbf{d}} < \tau_{\mathbf{u}}\}$$

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Standard Simulation Algorithm

- Set $X = 0$, $Z = 0$, and repeat N_1 times:
  - Start a replication at state $\mathbf{u}$, and stop it when it hits $\mathbf{d}$ or $\mathbf{u}$.
  - if it hits $\mathbf{d}$ do:
    - $X = X + 1$
    - $Z = Z + 1^2$
- $\widehat{\gamma}_0 = X/N_1$.
- $\widehat{\mathbb{V}}\{\widehat{\gamma}_0\} = (1/(N_1 - 1))(Z/N_1 - \widehat{\gamma}_0^2)$.

Conditional Monte Carlo

Fundamentals

Suppose that:

- $\mathbf{1}$ is the indicator of some event $\to \gamma = \mathbb{E}\{\mathbf{1}\}$.

Then,

- using some arbitrary random variable C, we have

$$\gamma = \mathbb{E}\{\mathbb{E}\{\mathbf{1} \mid C\}\}.$$

- Besides,

$$\begin{aligned} \mathbb{E}\{\mathbb{V}\{\mathbf{1} \mid C\}\} + \mathbb{V}\{\mathbb{E}\{\mathbf{1} \mid C\}\} &= \mathbb{V}\{\mathbf{1}\} \\ \text{So, } \mathbb{V}\{\mathbb{E}\{\mathbf{1} \mid C\}\} &\leq \mathbb{V}\{\mathbf{1}\} \end{aligned}$$

The key of Conditional Monte Carlo

The expectation of both, $\mathbb{E}\{\mathbf{1} \mid C\}$ and $\mathbf{1}$, is γ, but the variance of $\mathbb{E}\{\mathbf{1} \mid C\}$ is less than the variance of $\mathbf{1}$

Standard and Conditional Monte Carlo Estimators

- $\gamma = \mathbb{E}\{\mathbf{1}\}$: given the samples $\mathbf{1}^{(j)},\ j = 1, \ldots, N_1$,

$$\widehat{\gamma}_0 = \frac{1}{N_1} \sum_{j=1}^{N_1} \mathbf{1}^{(j)}$$

- $\gamma = \mathbb{E}\{\mathbb{E}\{\mathbf{1} \mid C\}\}$: given the samples $\mathbb{E}\{\mathbf{1} \mid C^{(j)}\},\ j = 1, \ldots, N_1$,

$$\widehat{\gamma}_1 = \frac{1}{N_1} \sum_{j=1}^{N_1} \mathbb{E}\{\mathbf{1} \mid C^{(j)}\}$$

  - Sample the values $C^{(j)}$
  - Calculate the corresponding $\mathbb{E}\{\mathbf{1} \mid C^{(j)}\}$

Basic Algorithm

The values to sample from, are known exactly

- $C = \{\mathbf{d}, k, \mathbf{u}\}$, k is any state (other than $\mathbf{d}$ or $\mathbf{u}$)
- X_C = the first state in C, hit by a replication that starts at $\mathbf{u}$

$$X_C = \begin{cases} \mathbf{d} & \text{w.p.} \quad p_{\mathbf{d}} \\ k & \text{w.p.} \quad p_k \\ \mathbf{u} & \text{w.p.} \quad p_{\mathbf{u}} \end{cases}$$

Then,

$$\mathbb{E}\{\mathbf{1} \mid X_C\} = \begin{cases} 1 & \text{w.p.} \quad p_{\mathbf{d}} \\ \gamma_k & \text{w.p.} \quad p_k \\ 0 & \text{w.p.} \quad p_{\mathbf{u}} \end{cases}$$

$$\widehat{\gamma}_1 = \frac{1}{N_1} \sum_{j=1}^{N_1} \mathbb{E}\{\mathbf{1} \mid X_C^{(j)}\}$$

Basic Algorithm – Many Intermediate States

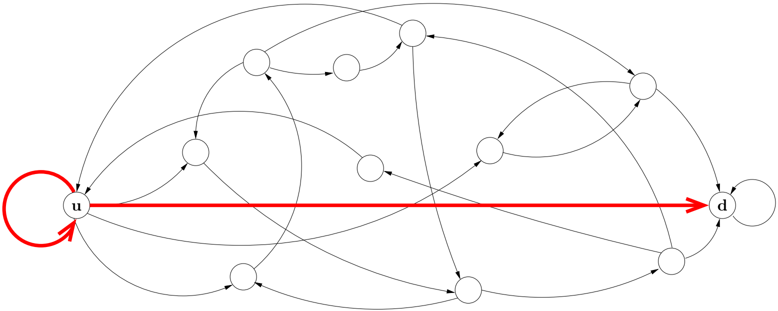
The values to sample from, are known exactly

- $C = \{\mathbf{d}, 1, 2, \ldots, n, \mathbf{u}\}$

$$\mathbb{E}\{\mathbf{1} \mid X_C\} = \begin{cases} \gamma_0 = 1 & \text{w.p.} \quad p_{\mathbf{d}} \\ \gamma_1 & \text{w.p.} \quad p_1 \\ \gamma_2 & \text{w.p.} \quad p_2 \\ \vdots & \\ \gamma_n & \text{w.p.} \quad p_n \\ 0 & \text{w.p.} \quad p_{\mathbf{u}} \end{cases}$$

$$\widehat{\gamma}_1 = \frac{1}{N_1} \sum_{j=1}^{N_1} \mathbb{E}\{\mathbf{1} \mid X_C^{(j)}\}$$

Standard Simulation
Graphical Illustration



Replications started at $\mathbf{u}$ can either:

- hit state $\mathbf{d}$ $\rightarrow$ accumulate a value of 1
- come back to state $\mathbf{u}$ $\rightarrow$ accumulate a value of 0

Estimate γ, as the average of all the accumulated values.

Basic Algorithm – Many Intermediate States

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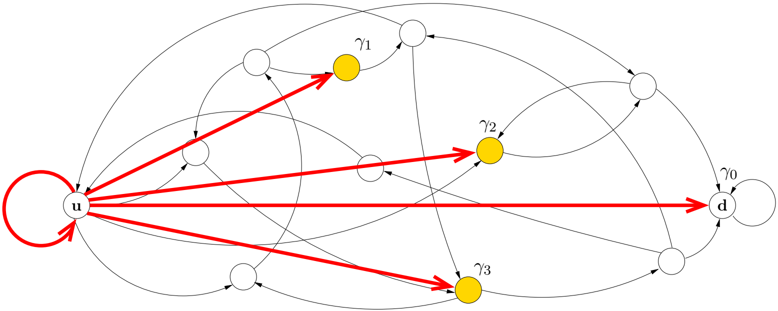


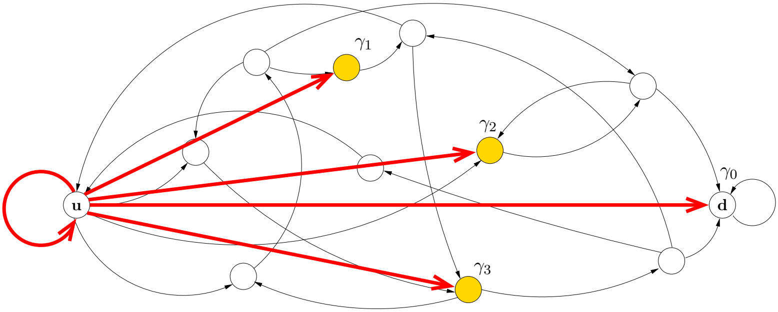
Replications started at $\mathbf{u}$ can either:

- hit state $\mathbf{d}$ $\rightarrow$ accumulate a value of $\gamma_0 = 1$
- hit one intermediate state $\rightarrow$ accumulate the corresponding γ_i
- come back to state $\mathbf{u}$ $\rightarrow$ accumulate a value of 0

Estimate γ, as the average of all the accumulated values.

Basic Algorithm – Many Intermediate States

Graphical Illustration



Remark

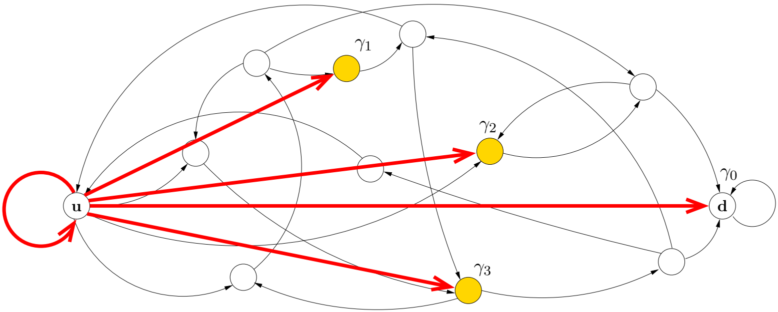
Conditional MC over many Intermediate States can be seen as a generalization of Crude or Standard Monte Carlo Simulation (average a set of real values instead of just 0s and 1s).

Conditional MC Simulation Algorithm

$$\begin{aligned}
\Gamma(\mathbf{d}) &= \gamma_0 = 1 \\
\Gamma(1) &= \gamma_1 \\
\Gamma(2) &= \gamma_2 \\
&\vdots \\
\Gamma(n) &= \gamma_n \\
\Gamma(\mathbf{u}) &= 0
\end{aligned}$$

- Set $X = 0$, $Z = 0$, and repeat N_1 times:
    - Start a replication at $\mathbf{u}$, and stop when it hits some $k \in C$.
    - $X = X + \Gamma(k)$.
    - $Z = Z + \Gamma(k)^2$.
- $\widehat{\gamma}_1 = X/N_1$.
- $\widehat{\mathbb{V}}\{\widehat{\gamma}_1\} = (1/(N_1 - 1))(Z/N_1 - \widehat{\gamma}_1^2)$.

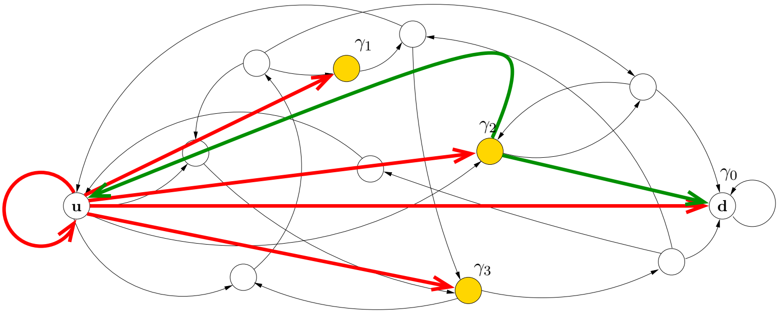
Conditional MC – Intermediate Estimations



Proposal

If the exact values $\{\gamma_1, \gamma_1, \ldots, \gamma_n\}$ are not available, standard estimators $\{\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots, \widehat{\gamma}_n\}$ can be used in place.

Conditional MC – Intermediate Estimations



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If the exact values $\{\gamma_1, \gamma_1, \ldots, \gamma_n\}$ are not available, standard estimators $\{\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots, \widehat{\gamma}_n\}$ can be used in place.

Conditional MC Simulation Algorithm
Intermediate Estimations

- Set $X = 0$, $Z = 0$, and repeat N_1 times:
  - Start a replication at $\mathbf{u}$, and stop when it hits some $k \in C$.
  - Set $Y = 0$ and repeat N_2 times:
    - Start a replication at state k, stop it when it reaches $\mathbf{d}$ or $\mathbf{u}$.
    - If the replication stops at $\mathbf{d}$, do $Y = Y + 1$.
  - $\widehat{\gamma}_k = Y/N_2$
  - $X = X + \widehat{\gamma}_k$
  - $Z = Z + \widehat{\gamma}_k^2$.
- $\widehat{\gamma}_2 = X/N_1$.
- $\widehat{\mathbb{V}}\{\widehat{\gamma}_2\} = (1/(N_1 - 1))(Z/N_1 - \widehat{\gamma}_2^2)$.

Variance Comparison

Standard:

$$\mathbb{V}\{\widehat{\gamma}_0\} = \frac{1}{N_1}\left(\gamma - \gamma^2\right) \;=\; \frac{1}{N_1}\left(\sum_{i=0}^{n} p_i \gamma_i - \gamma^2\right)$$

Pure Conditional MC – Exact Calculation:

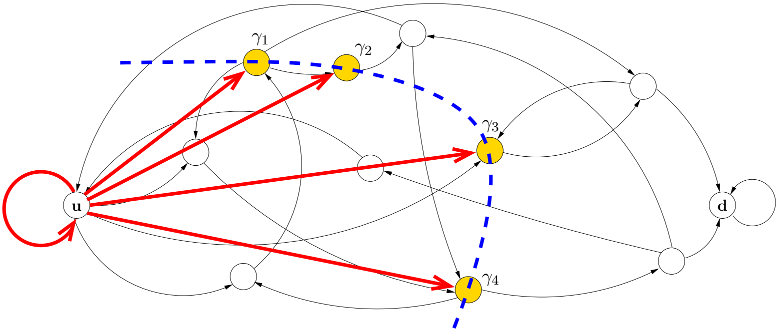
$$\mathbb{V}\{\widehat{\gamma}_1\} = \frac{1}{N_1}\left(\sum_{i=0}^{n} p_i \gamma_i^2 - \gamma^2\right)$$

Conditional MC – Intermediate Estimations:

$$\mathbb{V}\{\widehat{\gamma}_2\} = \frac{1}{N_1}\left(\sum_{k=0}^{n} p_k \gamma_k^2 - \gamma^2\right) + \frac{1}{N_1 N_2}\left(\gamma - \sum_{k=0}^{n} p_k \gamma_k^2\right)$$

The Particular Case of a Cut

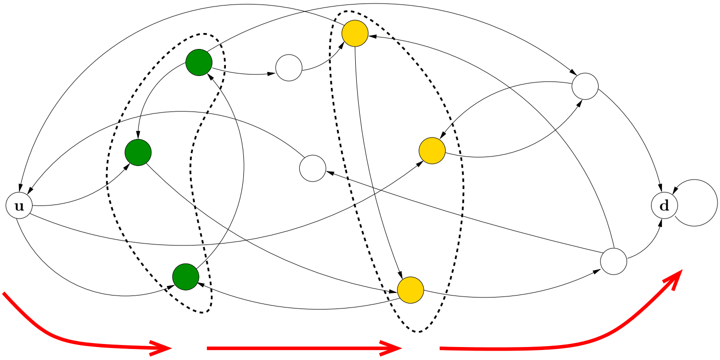
Conditional MC – Intermediate Estimations



- Trajectories started at $\mathbf{u}$ can never reach state $\mathbf{d}$.
- Our proposal corresponds to a typical Splitting process.

More than One Intermediate States

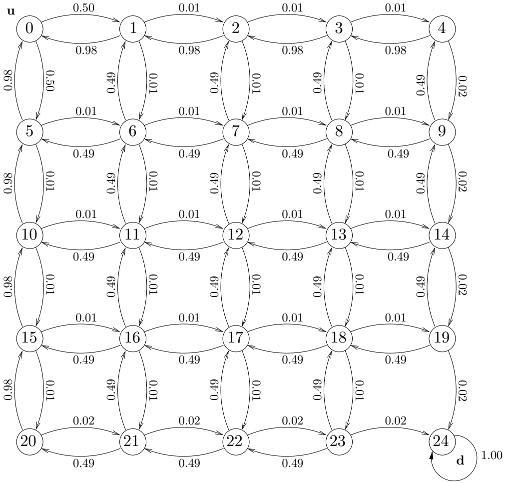
More than One Intermediate States



- Sets $\{C_1, C_2, \ldots, C_R\}$, not necessarily cuts.
- Proceed in stages: $\mathbf{u} \to C_1 \to \ldots C_R \to \mathbf{d}$
- Accuracy increase (at the expense of computational effort)

Model 1

The Markov Chain

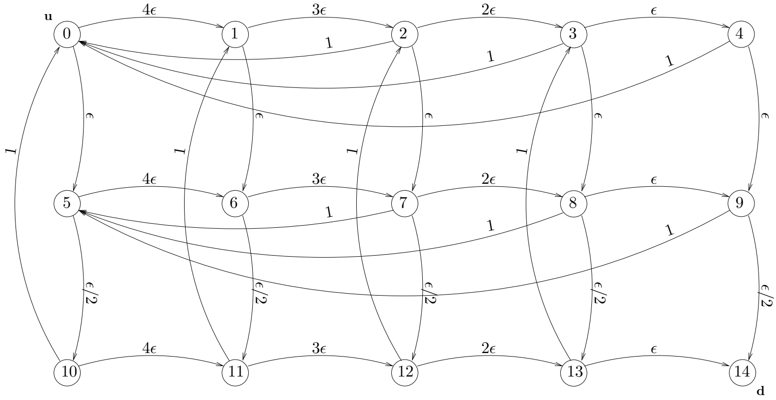


Model 1
Experimental Results

C_1	C_2	C_3	$\widehat{\gamma}$	$\mathbb{W}$ (*)
4–8–12–16–20	—	—	1.45e-12	44,695
2–6-10	14–18–22	—	1.47e-12	104,878
2–6–10	4–8–12–16–20	—	1.43e-12	53,937
4–8–12–16–20	14–18–22	—	1.44e-12	389,971
1–5	4–8–12–16–20	19–23	1.40e-12	691,542
2–6–10	4–8–12–16–20	14–18–22	1.47e-12	797,691
4–8–12–16–20	9–13–17–21	14–18–22	1.45e-12	443,232
9–13–17–21	14–18–22	19–23	1.50e-12	76,934

(*) $\mathbb{W} = (\widehat{\mathbb{V}}\{\widehat{\gamma}_0\} \times t_0)/(\widehat{\mathbb{V}}\{\widehat{\gamma}_2\} \times t_2)$

Model 2
The Markov Chain



Model 2

Experimental Results – $\epsilon = 0.001$

C_1	C_2	C_3	$\widehat{\gamma}$	$\mathbb{W}$ (*)
1–5	—	—	5.00e-09	2
2–6–10	—	—	6.00e-09	1
3–7–11	—	—	8.00e-09	1
4–8–12	—	—	5.98e-09	155
9–13	—	—	7.61e-09	516
2–6–10	3–7–11	—	9.00e-09	1
2–6–10	4–8–12	—	7.50e-09	119
2–6–10	9–13	—	6.77e-09	1,137
3–7–11	4–8–12	—	6.18e-09	146
3–7–11	9–13	—	6.47e-09	1,419
1–5	3–7–11	9–13	6.54e-09	311
2–6–10	3–7–11	4–8–12	4.92e-09	19
3–7–11	4–8–12	9–13	6.30e-09	5,265

(*) $\mathbb{W} = (\widehat{\mathbb{V}}\{\widehat{\gamma}_0\} \times t_0)/(\widehat{\mathbb{V}}\{\widehat{\gamma}_2\} \times t_2)$

Conclusions

- Conditional MC – Intermediate Estimations shows very high efficiency in several examples.
- In Markov Chains application, Multilevel Splitting is a particular case of Conditional MC – Intermediate Estimations (given the Importance Function, we can build the appropriate sets of intermediate states).
- Conditional MC – Intermediate Estimations has a flexibility that can be helpful in analyzing some families of complex models (failure propagation, component dependencies...)
- Conditional MC – Intermediate Estimations is of course quite easy to implement.

Current Work

- Exploration of different areas to find situations where the simplicity of the approach is relevant in practice.
- Test of the method on highly demanding models.
- Extension of the variance analysis to the case of multiple sets of intermediate states (not done yet).
- Looking for optimality results (parameter tuning).
- Comparison to other variance reduction methods.