Importance sampling in the neighborhood of a stable equilibrium point

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RESIM

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Problem of interest

Consider a process model such as

$$dX^{\varepsilon}(t) = b(X^{\varepsilon}(t))dt + \sqrt{\varepsilon}\sigma(X^{\varepsilon}(t))dW(t)$$

and assume $x = 0$ is a stable equilibrium for $\dot{\phi} = b(\phi)$ with domain of attraction $\mathcal{D}$.

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Example.

$$p_T^{\varepsilon} \doteq P\left\{X^{\varepsilon}(t) \text{ exits } \bar{\mathcal{D}} \text{ through } B \text{ before time } T \,|X^{\varepsilon}(0) = 0\right\},$$

where $B \subset \partial\bar{\mathcal{D}}$ and $\bar{\mathcal{D}} \subset \mathcal{D}$, and T is large.

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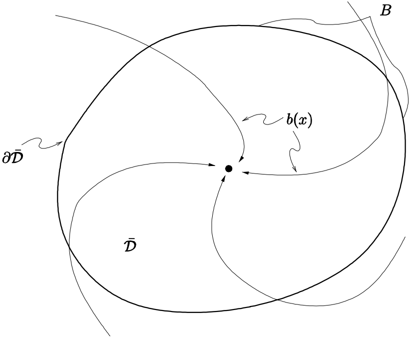

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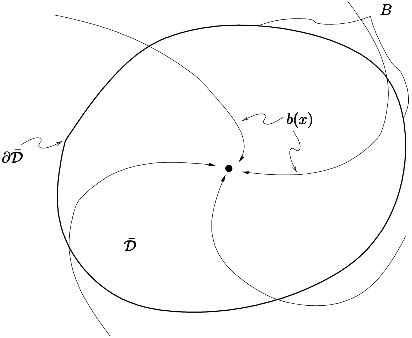

Also of interest: other process models, e.g., jump Markov. We focus on diffusion.

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Note the distinction in comparison to canonical problem studied in rare event simulation: the problem is time dependent and the stable point is in the domain of the event.

One possible approach

PDE/subsolutions approach to the design of importance sampling.

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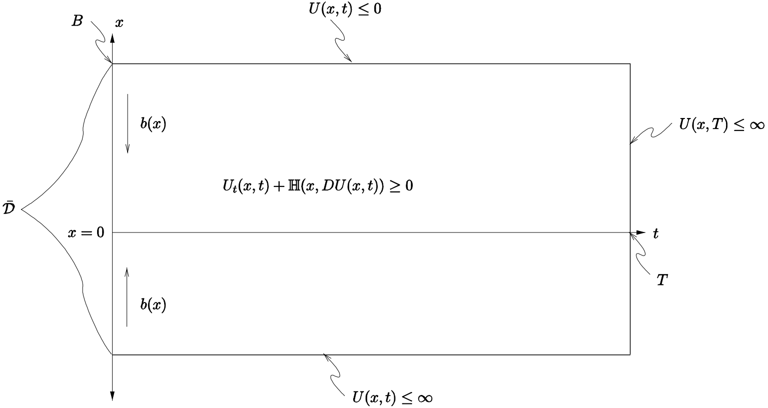
$$\mathbb{H}(x,p) = \langle p, b(x) \rangle - \frac{1}{2} \|\sigma(x)^T p\|^2$$

and let $U(x,t)$ satisfy



One possible approach

Theorem. Let $U(x,t)$ be a subsolution. Simulate

$$d\bar{X}^{\varepsilon}(t) = b(\bar{X}^{\varepsilon}(t))dt - \sigma(\bar{X}^{\varepsilon}(t))^{T} DU(\bar{X}^{\varepsilon}(t),t)dt + \sqrt{\varepsilon}\sigma(\bar{X}^{\varepsilon}(t))dW(t)$$

with $\bar{X}^{\varepsilon}(0) = 0$ and use samples

$$1\left\{\bar{\tau}^{\varepsilon} \leq T, \bar{X}^{\varepsilon}(\bar{\tau}^{\varepsilon}) \in B\right\} R^{\varepsilon}(\bar{X}^{\varepsilon}),$$

where $\bar{\tau}^{\varepsilon}$ is time of escape from $\bar{\mathcal{D}}$ and $R^{\varepsilon}(\bar{X}^{\varepsilon})$ is likelihood ratio of distribution of X^{ε} w.r.t $\bar{X}^{\varepsilon}$.

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where $\bar{\tau}^{\varepsilon}$ is time of escape from $\bar{\mathcal{D}}$ and $R^{\varepsilon}(\bar{X}^{\varepsilon})$ is likelihood ratio of distribution of X^{ε} w.r.t $\bar{X}^{\varepsilon}$. If $p^{\varepsilon}_{T} \approx e^{-\frac{1}{\varepsilon}\gamma_T}$, then

$$\limsup_{\varepsilon \to 0} \varepsilon \log E\left[1\left\{\bar{\tau}^{\varepsilon} \le T, \bar{X}^{\varepsilon}(\bar{\tau}^{\varepsilon}) \in B\right\} R^{\varepsilon}(\bar{X}^{\varepsilon})\right]^{2} \le -U(0,0) - \gamma_T.$$

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First question: Where to get good subsolutions?

One possible approach

When T is large the Freidlin-Wentsell quasipotential provides a *nearly optimal, time independent* subsolution. Let

$$Q(x) = \inf \left\{ \int_0^\tau \frac{1}{2} \left(\dot{\phi} - b(\phi) \right)^T \left[\sigma(\phi) \sigma(\phi)^T \right] \left(\dot{\phi} - b(\phi) \right) dt \right.$$

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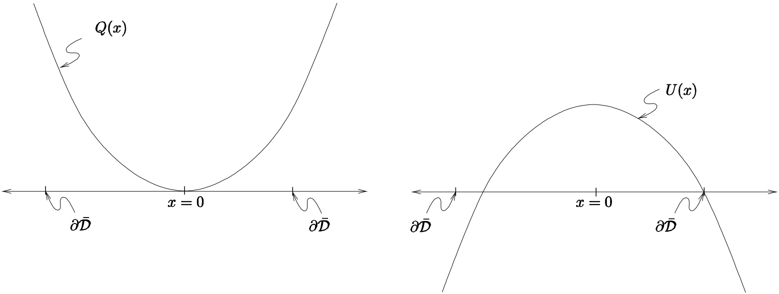
Then Q is available in explicit form for important classes of problems, and

$$U(x) = -Q(x) + \inf_{y \in B} Q(y)$$

is a subsolution with $U(0)$ slightly smaller than γ_T for T large.

One possible approach

Easy to check that $Q(0) = 0$ and $\mathbb{H}(x, -DQ(x)) = 0$. Then adding $\inf_{y \in B} Q(y)$ maximizes $U(0)$ subject to boundary conditions:



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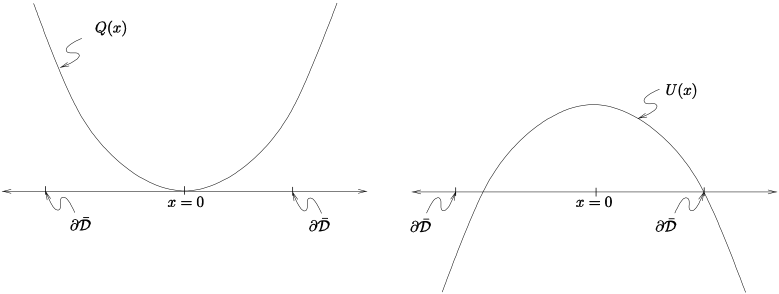

$$\gamma_T - U(0) = (\text{inf over } [0, T]) - (\text{inf over } [0, \infty)) \,.$$

Problem with the quasipotential subsolution

The induced importance sampling scheme has the proper decay rate $U(0) + \gamma_T$, but performance not as good as one expects. Reason for poor performance: dual exponential scalings, one in $1/\varepsilon$, one in T.

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Example. 1D Gauss-Markov with $\bar{\mathcal{D}} = [-1, 1], B = \partial\bar{\mathcal{D}}, c = 1, \bar{\sigma} = 1$:

$$dX^{\varepsilon}(t) = -cX^{\varepsilon}(t)dt + \sqrt{\varepsilon}\bar{\sigma}dW(t).$$

Relative errors:

T vs ε	0.25	0.5	1	1.5	2.5	10	14	18
0.20	91	7	2	1	1	10	51	179
0.16	253	10	2	1	1	10	48	139
0.13	748	16	3	1	1	9	48	378
0.11	1594	26	3	1	1	10	42	272
0.09	–	49	4	2	1	9	43	357
0.07	–	127	5	2	1	8	47	251
0.05	–	714	8	2	1	8	42	145

Problem with the quasipotential subsolution

A non-asymptotic representation for the 2nd moment. Given control process v let $\hat{X}^{\varepsilon}(0) = 0$ and

$$d\hat{X}^{\varepsilon}(t) = -c\hat{X}^{\varepsilon}(t)dt + \bar{\sigma}DU(\hat{X}^{\varepsilon}(t))dt + v(t)dt + \sqrt{\varepsilon}\bar{\sigma}dW(t).$$

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Based on a representation theorem (Boué-D, *Annals of Probability*, 1998) we have the *non-asymptotic* bound

$$\begin{aligned}&-\varepsilon\log E\left[1\left\{\bar{\tau}^{\varepsilon}\leq T,\bar{X}^{\varepsilon}(\bar{\tau}^{\varepsilon})\in B\right\}R^{\varepsilon}(\bar{X}^{\varepsilon})\right]^{2}\\ &\quad=\inf_{v}E\left[\frac{1}{2}\int_{0}^{\hat{\tau}^{\varepsilon}}\left|v(s)\right|^{2}ds-\int_{0}^{\hat{\tau}^{\varepsilon}}\left|DU(\hat{X}^{\varepsilon}(s))\right|^{2}ds+\infty 1\left\{\hat{\tau}^{\varepsilon}>T\right\}\right].\end{aligned}$$

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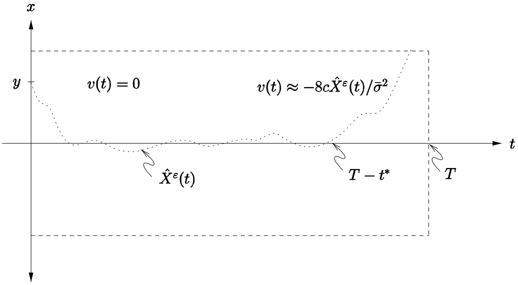

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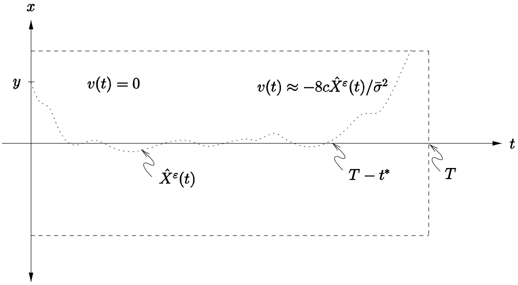

Using the control above one obtains

$$E\left[\frac{1}{2}\int_0^{\hat{\tau}^\varepsilon}|v(s)|^2\,ds-\int_0^{\hat{\tau}^\varepsilon}|DU(X^\varepsilon(s))|^2\,ds+\infty 1\left\{\hat{\tau}^\varepsilon>T\right\}\right]\leq-\varepsilon C_1[T-t^*]+C_2,$$

and thus

$$E\left[1\left\{\bar{\tau}^\varepsilon\leq T,\bar{X}^\varepsilon(\bar{\tau}^\varepsilon)\in B\right\}R^\varepsilon(\bar{X}^\varepsilon)\right]^2\geq e^{C_1[T-t^*]}e^{-\frac{1}{\varepsilon}C_2}.$$

Resolution of the problem

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$$U_t^\varepsilon(x,t) + \langle DU^\varepsilon(x,t), b(x)\rangle - \frac{1}{2}\left\|\sigma(x)DU^\varepsilon(x,t)\right\|^2 + \frac{\varepsilon}{2}\mathrm{tr}\left[\sigma(x)\sigma^T(x)D^2U^\varepsilon(x,t)\right] = 0.$$

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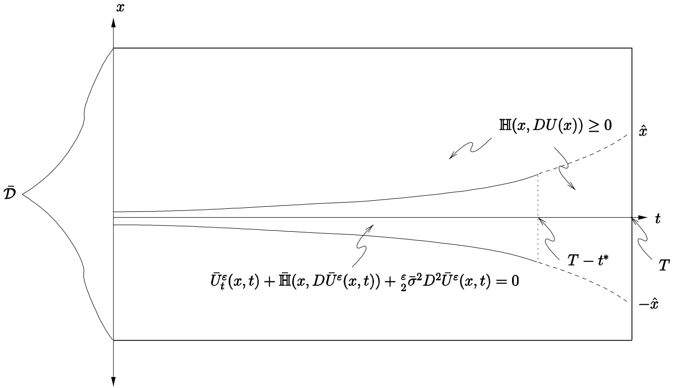
- Combine quasipotential subsolution away from rest point with explicit solution to an approximating problem near rest point (linear quadratic regulator).
- Construct time and ε-dependent subsolution with bounds that are (proveably) uniform in T.

Construction of a time-dependent subsolution

Basic form of the partition:



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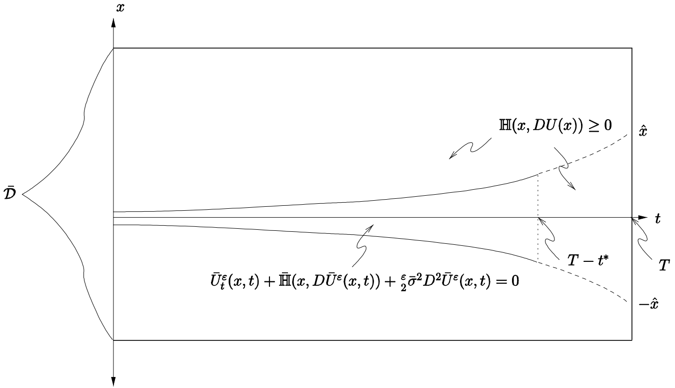

$\bar{\mathbb{H}}$ is the Hamiltonian of the *linearized* system. A heuristic motivation: better approximation to the zero-variance scheme for the time dependent problem, which previously was poorly approximated near $x = 0$.

Construction of a time-dependent subsolution

Consider one-dimensional problem, with linearization $c = b^{'}(0)$, $\bar{\sigma} = \sigma(0)$. For $0 \leq x \leq e^{c(t-T)}$, let $\bar{U}^{\varepsilon}(x,t)$ be solution to linear quadratic regulator with terminal condition $\bar{U}^{\varepsilon}(x,T) = M\left(x - \hat{x}\right)^{2}/2 + U\left(\hat{x}\right)$.

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$$\begin{aligned} \bar{U}^{\varepsilon}(x,t) &= a^M(t)\left(x - \hat{x}e^{-c(t-T)}\right)^2 + U(\hat{x}) \\ &\quad -\frac{\varepsilon}{2}\log\left((2c/M+\bar{\sigma}^2) - \bar{\sigma}^2 e^{2c(t-T)}\right), \\ a^M(t) &= \frac{ce^{2c(t-T)}}{(2c/M+\bar{\sigma}^2) - \bar{\sigma}^2 e^{2c(t-T)}} > 0. \end{aligned}$$

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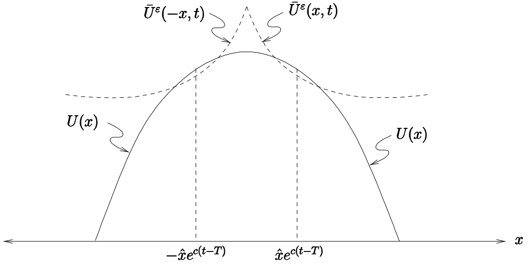
&\quad -\frac{\varepsilon}{2}\log\left(\left(2c/M + \bar{\sigma}^{2}\right) - \bar{\sigma}^{2}e^{2c(t-T)}\right), \\
a^{M}(t) &= \frac{ce^{2c(t-T)}}{\left(2c/M + \bar{\sigma}^{2}\right) - \bar{\sigma}^{2}e^{2c(t-T)}} > 0.
\end{aligned}
$$

For $-e^{c(t-T)} \leq x \leq 0$ have $\bar{U}^{\varepsilon}(x,t) = \bar{U}^{\varepsilon}(-x,t)$. Here $\hat{x}$ and M parameters used in the design to control region where $\bar{U}^{\varepsilon}(x,t)$ will define subsolution.

Construction of a time-dependent subsolution

Number of details in merging various functions. *Most important issue:* need to assemble time dependent subsolution as minima of several functions.



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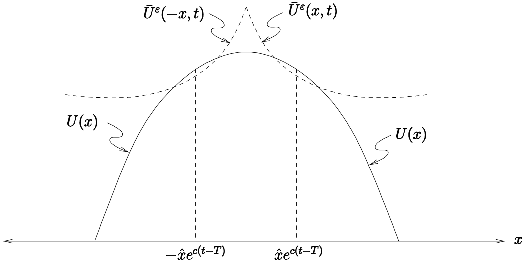

Second derivatives are $-\infty$ at handoff between functions. Causes problems and needs to be avoided.

Construction of a time-dependent subsolution

We use the *exponential mollification*

$$\bar{U}^{\delta}(t,x) = -\delta \log \left(\sum_{i=1}^{K} e^{-\frac{1}{\delta} F^{i}(t,x)} \right)$$

for various F^i ($= \bar{U}^{\varepsilon}(x,t), \bar{U}^{\varepsilon}(-x,t)$, etc.).

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First introduced for importance sampling in (D. and Wang, *Mathematics of Operations Research*, 2007), and to obtain non-aymptotic bounds in (Blanchet, Glynn and Leder, *TOMACS*, 2011).

Construction of a time-dependent subsolution

Lemma. We have the bounds

$$\wedge_{i=1}^{K} F^i(t,x) \geq \bar{U}^\delta(t,x) \geq \wedge_{i=1}^{K} F^i(t,x) - \delta \log K.$$

Suppose

$$F_t^i(x,t) + \left\langle DF^i(x,t), b(x) \right\rangle - \frac{1}{2} \left\| \sigma(x) DF^i(x,t) \right\|^2 + \frac{\varepsilon}{2} \mathrm{tr} \left[\sigma(x)\sigma^T(x) D^2 F^i(x,t) \right] \\ \geq \gamma^i(x,t,\varepsilon)$$

and let

$$\rho^i(x,t;\delta) = e^{-\frac{1}{\delta}F^i(t,x)} \Big/ \sum_{j=1}^{K} e^{-\frac{1}{\delta}F^j(t,x)}.$$

Then for $\delta \geq \varepsilon$

$$\bar{U}_t^\delta(x,t) + \left\langle D\bar{U}^\delta(x,t), b(x) \right\rangle - \frac{1}{2} \left\| \sigma(x) D\bar{U}^\delta(x,t) \right\|^2 + \frac{\varepsilon}{2} \mathrm{tr} \left[\sigma(x)\sigma^T(x) D^2 \bar{U}^\delta(x,t) \right] \\ \geq \sum_{i=1}^{K} \rho^i(x,t;\delta)\gamma^i(x,t,\varepsilon).$$

Properties of the time-dependent subsolution

Using the mollification with $\delta = \varepsilon$ to construct the subsolution and then applying a verification argument, we get the *non-asymptotic* bound

$$\left(1 - \frac{\varepsilon\bar{\sigma}^2}{4c\hat{x}^2}e^{2ct^*}\right)\left[U(0) - \frac{4c}{\bar{\sigma}^2}\hat{x}^2 e^{-2ct^*}\right] - \varepsilon\log 2$$
$$+ \frac{\varepsilon}{4\bar{\sigma}}\left[\log\left(\frac{2c}{M} + \sigma^2 - \frac{\sigma^2 c}{2c + M\bar{\sigma}^2}\right) - \log\left(\frac{2c}{M} + \sigma^2\right)\right]$$
$$+ \frac{\varepsilon}{4}\log\left(\frac{c}{2\left(2c + M\bar{\sigma}^2\right)}\right).$$

Properties of the time-dependent subsolution

Previous example with $M = 1$, $t^* = 0.9$, $\delta = \varepsilon$, and $\hat{x} \approx .5$:

T vs ε	1	1.5	2.5	10	14	18
0.20	2	2	2	1	1	1
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Ongoing work

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- Extension to the non-Gaussian case.