Efficient Monte Carlo Algorithms for Computing High Quantiles

Motivation from Insurance

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Outline

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General context

- Solvency II - the new regulatory framework
- Internal models for risk assessment
- Solvency principle: leave the company “as is” for one year, then the value of assets

Insurance portfolio

- Assets (fixed income, stocks, real estate)
- Liabilities (claim payments, reserves)
- Solvency capital requirement: $\mathrm{VaR}_{0.005}(A_1 - L_1) < 0$ or equivalently $F^{-1}_{e^{-r_1}(L_1 - A_1)}(0.995) < 0$, where

$$e^{-r_1}(L_1 - A_1) = e^{-r_1} \sum_{k=1}^{n} (E[C_k \mid \mathcal{F}_1] - E[C_k]) e^{-(r_{k-1} + \Delta r_{k-1})(k-1)}$$

$$+ \text{discounted loss from assets.}$$

Claims Triangle: observed claims

	Development year					
Origin	0	1	2	$\cdots$	$n-1$	n
$-n-1$	$C_{-n-1,0}$	$C_{-n-1,1}$	$C_{-n-1,2}$	$\cdots$	$C_{-n-1,n-1}$	$C_{-n-1,n}$
$-n$	$C_{-n,0}$	$C_{-n,1}$	$C_{-n,2}$	$\cdots$	$C_{-n,n-1}$	
$\vdots$	$\vdots$	$\vdots$				
-2	$C_{-2,0}$	$C_{-2,1}$				
-1	$C_{-1,0}$					
0						

Table: The observed upper triangle of paid claims. $C_{-i,j}$ is cumulative claim amount for claims occurring in period $-i$ and paid before time $-i+j$.

Claims Triangle: unobserved claims

	Development year					
Origin	0	1	2	$\cdots$	$n-1$	n
$-n-1$						
$-n$						$C_{-n,n}$
$-n+1$					$C_{-n+1,n-1}$	$C_{-n+1,n}$
$\vdots$					$\vdots$	$\vdots$
-1		$C_{-1,1}$	$C_{-1,2}$	$\cdots$	$C_{-1,n-1}$	$C_{-1,n}$
0	$C_{0,0}$	$C_{0,1}$	$C_{0,2}$	$\cdots$	$C_{0,n-1}$	$C_{0,n}$

Table: The unobserved triangle of outstanding claims to be predicted.

Need for Simulation Algorithms

- Many dependent risk factors
- Rare events of interest
- Need decent approximation of the tail of the distribution (not only quantile estimates)
- Possible reduction of computational cost by efficient sampling

Outline

Main message

Designing efficient importance sampling algorithms for rare event probabilities is a good starting point for computing (tail) risk measures efficiently.

Risk measures

Value-at-Risk and Expected Shortfall

Consider a random variable $X \sim F_X$, representing the net worth at time 1. Let $L = -e^{-r_1} X$ be the discounted loss.

- Value-at-Risk (quantile):

$$\mathrm{VaR}_p(X) = F_L^{-1}(1-p) = \inf\{x : F_L(x) \geq 1-p\},$$

- Expected shortfall:

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_u(X)\,du$$

where p is small; say 0.005.

Outline

Value-at-Risk

- Generate $L_1, \ldots, L_N$ independently from the sampling distribution G.
- Form the weighted empirical distribution/tail

$$\mathbb{G}_N(\cdot) = \frac{1}{N} \sum_{j=1}^{N} \frac{dF}{dG}(L_j) \delta_{L_j}(\cdot),$$

$$\mathbb{G}_N(x) = \frac{1}{N} \sum_{j=1}^{N} \frac{dF}{dG}(L_j) I\{L_j > x\}, \quad x \in R.$$

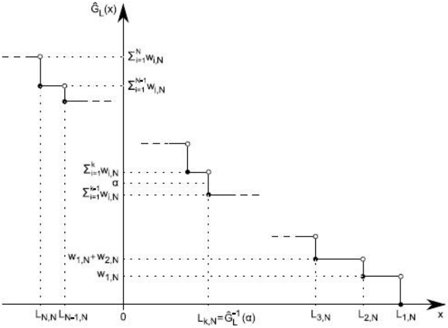
VaR continued

- Estimate VaR_p by $\mathbb{G}_N^{-1}(p)$ where

$$\mathbb{G}_N^{-1}(p) = \inf\{x : \mathbb{G}_N(x) \le p\} = \{\text{picture}\} = L_{k,N},$$
$$k = \inf\{m : w_1 + \cdots + w_m \ge p\},$$

and $w_j = \frac{dF}{dG}(L_{j,N})$.

Illustration



From diploma thesis of P. Müller (KTH/ETH)

Risk measures by importance sampling

Expected Shortfall

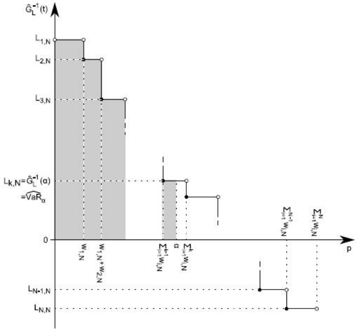
Computing Expected Shortfall by importance sampling:

- Generate $L_1, \ldots, L_N$ independently from G and form $\mathbb{G}_N$.
- Compute the estimate

$$\frac{1}{p}\int_0^p \mathbb{G}_N^{-1}(u)du = \{\text{picture}\}$$

$$= \frac{1}{p}\Big(\sum_{j=1}^{k-1} w_j L_{j,N} + (p - \sum_{j=1}^{k-1} w_j) L_{k,N}\Big).$$

Illustration



From diploma thesis of P. Müller (KTH/ETH)

Outline

Asymptotic theory

The large sample limit

- Empirical process theory (CLTs as $N \to \infty$)
- Large deviations theory (extensions of Sanov's theorem) as $N \to \infty$

Empirical processes

- Suppose the empirical measures satisfies a CLT

$$\sqrt{N}(\mathbb{G}_N - F) \xrightarrow{w} Z,$$

  where Z is centered Gaussian,
- ... and the delta-method implies

$$\sqrt{N}(\phi(\mathbb{G}_N) - \phi(F)) \xrightarrow{w} \phi'_F(Z),$$

- then we can use the asymptotic variance $\mathrm{Var}(\phi'_F(Z))$ to measure efficiency.

Objective: select sampling distribution G_n to get $\mathrm{Std}(\phi'_F(Z))$ of roughly the size of $\phi(F)$.

Empirical processes and importance sampling

The CLT

Let G be the sampling distribution and $w(\cdot) = \frac{dF}{dG}(\cdot)$.

- Let $\mathcal{F}_a$ be the indicator functions $x \mapsto I\{x > t\}$, $t \geq a$.
- Then, the weighted empirical measure

$$\mathbb{G}_N(f) = \frac{1}{N}\sum_{j=1}^{N} \frac{dF}{dG}(L_j) f(L_j) = \tilde{\mathbb{F}}_N(wf),$$

where $\tilde{\mathbb{F}}_N$ is the empirical measure based on L_j, generated from G;

$$\tilde{\mathbb{F}}_N(f) = \frac{1}{N}\sum_{j=1}^{N} f(L_j).$$

Empirical processes and importance sampling

The CLT

- We have a CLT: $\sqrt{N}(\mathbb{G}_N(f) - F(f)) \xrightarrow{w} Z$ in $L^\infty(\mathcal{F}_a)$ precisely if $w\mathcal{F}_a$ is a G-Donsker class.
- A sufficient condition for $w\mathcal{F}_a$ to be G-Donsker is that $\mathcal{F}_a$ is G-Donsker and $E_F[w(X_j)I\{X_j > a\}] < \infty$.

Empirical processes and importance sampling

Identifying the limit process

The limiting process Z is centered Gaussian with covariance function

$$\begin{aligned}\varrho(x,y) &= N\,\mathrm{Cov}(\mathbb{G}_N(x),\mathbb{G}_N(y))\\ &= E_G[\mathbb{G}_N(x)\mathbb{G}_N(y)] - E_G[\mathbb{G}_N(x)]E_G[\mathbb{G}_N(y)]\\ &= \dots\\ &= E_F[w(X)I\{X>y\}] - F(x)F(y),\end{aligned}$$

when $y \geq x \geq a$.

Outline

VaR by importance sampling

CLT

Theorem Let Z be centered Gaussian with covariance ϱ. Suppose F has a continuous density $f > 0$ on the interval $[F^{-1}(q) - \epsilon, F^{-1}(p) + \epsilon]$, for $0 < p < q < 1$ and $\epsilon > 0$. Then

$$\sqrt{N}(\mathbb{G}_N^{-1} - F^{-1}) \xrightarrow{w} \frac{Z \circ F^{-1}}{f \circ F^{-1}}, \quad \text{in } L^\infty[p, q],$$

where the right-hand side refers to the random function $u \mapsto \frac{Z(F^{-1}(u))}{f(F^{-1}(u))}$.

ES by importance sampling

CLT

Expected shortfall is given by

$$ES_{1-p}(X) = \frac{1}{p}\int_0^p F^{-1}(u)du =: \gamma_p(F^{-1}).$$

It is estimated by $\gamma_p(\mathbb{G}_N^{-1})$.

ES by importance sampling

CLT

Theorem Assume the hypotheses in the previous theorem and in addition that $\rho(x,x) = o([f(x)/F(x)]^2)$ and

$$\int_{F^{-1}(p)}^{\infty} \int_{F^{-1}(p)}^{\infty} \varrho(x,y) dxdy < \infty.$$

Then

$$\sqrt{N}(\gamma_p(\mathbb{G}_N^{-1}) - \gamma_p(F^{-1})) \xrightarrow{\text{w}} \gamma_p\left(\frac{Z \circ F^{-1}}{f \circ F^{-1}}\right) = \frac{1}{p}\int_0^p \frac{Z(F^{-1}(u))}{f(F^{-1}(u))} du.$$

Outline

Robbins-Monro algorithm

- Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and decreasing.
- Look for x^* solving $f(x^*) = 0$.
- Robbins-Monro Algorithm:
  - Take $X_n = X_{n-1} + \rho_n Y_n$ where $E[Y_n \mid X_{n-1}] = f(X_{n-1})$ and

$$\sum \rho_n = \infty, \quad \sum \rho_n^2 < \infty.$$

- *Example:* Take $f(x) = \overline{F}(x) - p$ to search for a quantile.

Outline

1 Quantifying Insurance Risk
- Risk and Solvency

2 Computing Risk Measures by Simulation
- Risk measures
- Computation by importance sampling

3 Empirical processes and Large deviations
- Empirical processes and importance sampling
- Risk measures by importance sampling

4 Random iterative methods
- Robbins-Monro
- Bayesian Root Finding

A Bayesian Root Finding Method

(Horstein, '63; Waeber, Frazier, and Henderson, '11)

- Let $f : (0,1) \to \mathbf{R}$ be continuous and decreasing.
- Look for x^* solving $f(x^*) = 0$.
- Can evaluate $f(x)$ with error and observe $Y(x)$ where

$$Y(x) = \begin{cases} 1 & \text{w.p.} \quad q \quad \text{if } x < x^*, \\ -1 & \text{w.p.} \quad 1-q \quad \text{if } x < x^*, \\ 1 & \text{w.p.} \quad 1-q \quad \text{if } x > x^*, \\ -1 & \text{w.p.} \quad q \quad \text{if } x > x^*. \end{cases}$$

- Algorithm:
  - Let π_0 be a prior on x^*.
  - Measure at $x_1 = \text{median}(\pi_0)$.
  - Observe $Y_1(x)$ and compute posterior $\pi_1(x^* \mid Y_1(x))$.
  - Take $x_2 = \text{median}(\pi_1)$ and iterate.

A quantile estimation test problem

Heavy-tailed random walk

- Let $X = S_k = Z_1 + \cdots + Z_k$ a heavy-tailed random walk (reg. var.)
- Then $\overline{F}(x) = x^{-\alpha} L(x)$ where L is slowly varying
- Have efficient importance sampler for computing $\overline{F}(x)$.
- How to select thresholds $x_1, x_2, \ldots$ to obtain
  1. a good estimate of $x_p = F^{-1}(p)$ with p close to 1?
  2. a good approximation of $\overline{F}$ above x_p?

A Normal Bayesian Algorithm

(with C. Mercadier)

- Aim to estimate x_p by first estimating x_p^α.
- Start with prior on x_p^α given as $N(x_1^\alpha, v_1^2)$.
- Compute estimate $\widehat{p}_1$ of $\overline{F}(x_1)$ by I.S. based on sample size m.
- $\widehat{p}_1$ is approx $N(\overline{F}(x_1), \sigma_1^2/m)$.
- Using $\overline{F}(cx) \approx c^{-\alpha}\overline{F}(x)$ with $c = x_1/x_p$, $x = x_p$ it follows that $\widehat{p}_1$ is approx $N(p(x_p/x_1)^\alpha, \sigma_1^2/m)$.

A Normal Bayesian Algorithm

(with C. Mercadier)

- Then the posterior of x_p^α is normal with

$$\text{mean} = x_1^\alpha + \frac{mpv_1^2}{x_1^{2\alpha}\sigma_1^2 + mp^2v_1^2} x_1^\alpha(\widehat{p}_1 - p)$$

$$\text{var} = v_1^2 - \frac{mpv_1^2}{x_1^{2\alpha}\sigma_1^2 + mp^2v_1^2} pv_1^2.$$

- Iterating the algorithm gives the following updates:

$$x_{n+1}^\alpha = x_n^\alpha + \frac{mpv_n^2}{x_n^{2\alpha}\sigma_n^2 + mp^2v_n^2} x_n^\alpha(\widehat{p}_n - p)$$

$$v_{n+1}^2 = v_n^2 - \frac{mpv_n^2}{x_n^{2\alpha}\sigma_n^2 + mp^2v_n^2} pv_n^2.$$

Summary

- Efficient simulation is relevant in insurance (and finance) application, when building models for risk assessment.
- Efficient importance algorithms for computing rare-event probabilities is a good starting point for computing risk measures.
- Theoretical analysis is possible by means of empirical process theory (CLT/LDP)
- Adapted importance sampling procedures needed to find the appropriate tail region.

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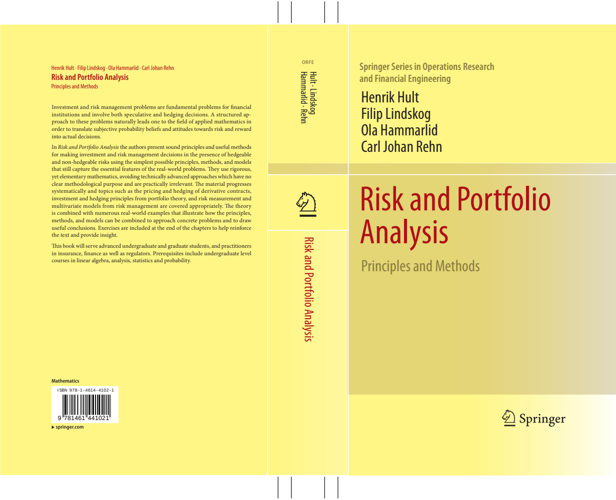
Henrik Hult · Filip Lindskog · Ola Hammarlid · Carl Johan Rehn
Risk and Portfolio Analysis
Principles and Methods
Investment and risk management problems are fundamental problems for financial institutions and involve both speculative and hedging decisions. A structured approach to these problems naturally leads one to the field of applied mathematics in order to translate subjective probability beliefs and attitudes towards risk and reward into actual decisions.
In Risk and Portfolio Analysis the authors present sound principles and useful methods for making investment and risk management decisions in the presence of hedgeable and non-hedgeable risks using the simplest possible principles, methods, and models that still capture the essential features of the real-world problems. They use rigorous, yet elementary mathematics, avoiding technically advanced approaches which have no clear methodological purpose and are practically irrelevant. The material progresses systematically and topics such as the pricing and hedging of derivative contracts, investment and hedging principles from portfolio theory, and risk measurement and multivariate models from risk management are covered appropriately. The theory is combined with numerous real-world examples that illustrate how the principles, methods, and models can be combined to approach concrete problems and to draw useful conclusions. Exercises are included at the end of the chapters to help reinforce the text and provide insight.
This book will serve advanced undergraduate and graduate students, and practitioners in insurance, finance as well as regulators. Prerequisites include undergraduate level courses in linear algebra, analysis, statistics and probability.
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